Method of Separation of Variables for the Solution of One-Dimensional Wave Equation

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Abstract

In this research paper, we first introduce the basic concepts of Fourier series, derivation of the Euler formulas, Fourier cosine and sine series. And then basic concepts of partial differential equations, order, linearity, principle of superposition are presented. Finally, the method of separating variables for solving a vibrating elastic string problem is also described.

Introduction

Periodic phenomena occurs quite frequently-think of motors, rotating machines, sound waves, the motion of the earth, and the heart under normal conditions. In such a case, it is an important practical problem to represent the corresponding periodic functions in terms of simple periodic functions, namely, cosine and sine. These representations will be series? called Fourier series.

Section-I is primarily concerned with Fourier series.

Section-II is concerned with the most important partial differential equations of physics and engineering.

In the last section, we shall introduce one of the most common and elementary methods, called the method of separation of variables, for solving initial-boundary value problems. The class of problems for which this method is applicable which contains a wide range of problems of mathematical physics, applied mathematics, and engineering sciences.

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1. Basic Concepts of Fourier Series

1.1 Fourier Series

Fourier series are infinite series that represent periodic functions in terms of cosines and sines. To define Fourier series, we first need some background material. A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x, except possibly at some points,and if there is some positive number *p*, called a period of $f(x)$, such that

$$f(x+p)=f(x) \quad \text{for all } x. \tag{1}$$

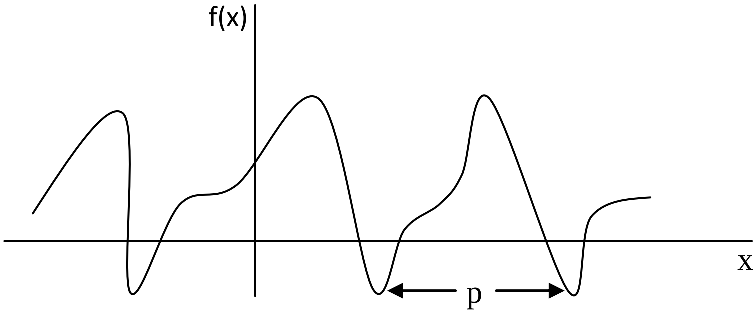


Figure 1 Periodic Function of Period p

The smallest positive period is often called the fundamental period.

If *f(x)* has period *p*, it is also has the period 2*p* because (1) implies $f(x+2p)=f([x+p]+p)=f(x+p)=f(x)$, etc.; thus for any integer $n=1,2,3,...,$

$$f(x+np)=f(x) \text{ for all } x. \tag{2}$$

Furthermore if $f(x)$ and $g(x)$ have period *p*, then $af(x)+bg(x)$ with any constants *a* and *b* also has the period *p*.

Our problem in the first few sections of this chapter will be the representation of various functions $f(x)$ of period 2π in terms of the simple functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, ..., \cos nx, \sin nx, ... \tag{3}$$

All these functions have the period 2π. They form the so-called trigonometric system. Figure 2 shows the first few of them (except for the constant 1, which is periodic with any period).

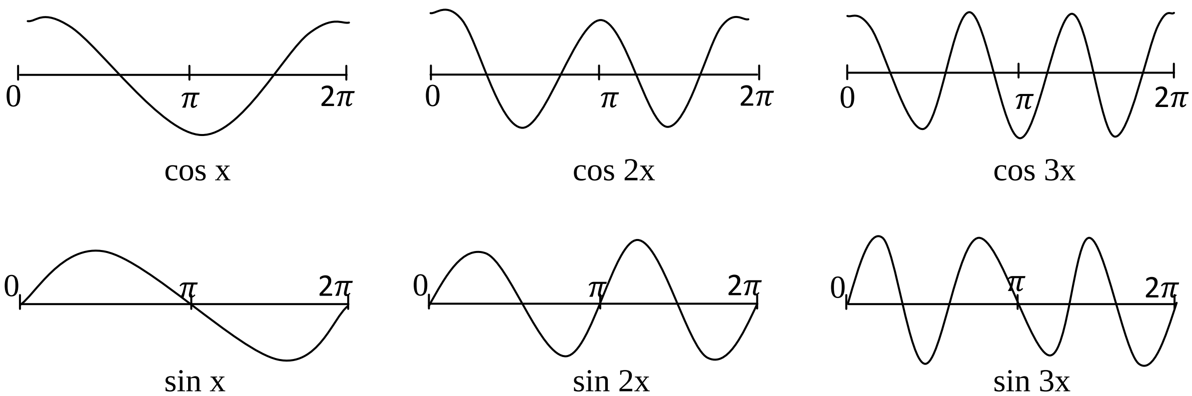


Figure-2 Cosine and Sine Functions Having The Period 2*π* (The first few members of the trigonometric system (3), except for the constant 1)

The series to be obtained will be a trigonometric series, that is, a series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \tag{4}$$

$a_0, a_1, b_1, a_2, b_2, \ldots$ are constants, called the coefficients of the series. We see that each term has the period 2π. Hence if the coefficients are such that the series converges, its sum will be a function of period 2π.

Now suppose that $f(x)$ is a given function of period 2π and is such that it can be represented by a series(4), that is,(4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \tag{5}$$

and call (5) the Fourier series of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called Fourier coefficients of $f(x)$, given by the Euler formulas

$$\text{(i)} \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$\text{(ii)} \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad n = 1, 2, \ldots \tag{6}$$

$$\text{(iii)} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \qquad n = 1, 2, \ldots.$$

1.2 Example of Fourier Series

We can find the Fourier coefficients of the periodic function $f(x)$ in Figure 3. The formula is

$$f(x) = \begin{cases} -k & if \quad -\pi < x < 0 \\ k & if \quad 0 < x < \pi \end{cases} \text{and} \quad f(x + 2\pi) = f(x). \tag{7}$$

From $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$, we obtain $a_0 =$ 0.This can also be seen without integration, since the area under the curve of *f(x)* between $-\pi$ and π (taken with a minus sign where *f(x)* is negative) is zero. From $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$, $n = 1, 2, \ldots,$ we obtain the coefficients $a_1, a_2, \ldots$ of the cosine terms. Since $f(x)$ is given by two expressions, the integrals from $-\pi$ to π split into two integrals:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx dx + \int_{0}^{\pi} k\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$

because $\sin nx = 0$ at $-\pi$,0, and π for all $n = 1, 2, ...$. We see that all these cosine coefficients are zero. That is, the Fourier series of (7) has no cosine terms, just sine terms, it is a Fourier sine series with coefficients $b_1, b_2, ...$ obtained from

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx,\ n = 1, 2, ...;$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi} k\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\frac{\cos nx}{n}\Big|_{0}^{\pi}\right].$$

Since $\cos(-\alpha) = \cos(\alpha)$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc; in general,

$$\cos n\pi = \begin{cases} -1 & for\ \ odd\ \ n, \\ 1 & for\ \ even\ \ n, \end{cases} \qquad \text{and thus } 1 - \cos n\pi = \begin{cases} 2 & for\ \ odd\ \ n, \\ 0 & for\ \ even\ \ n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi},\ b_2 = 0,\ b_3 = \frac{4k}{3\pi},\ b_4 = 0,\ b_5 = \frac{4k}{5\pi}, \cdots.$$

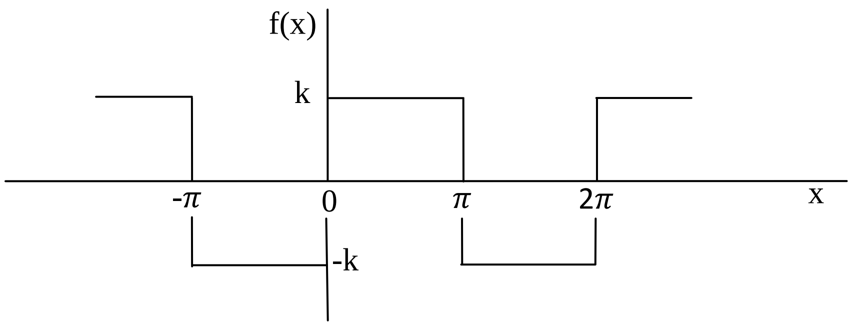


Figure 3 Given Function f(x) (Periodic Rectangular Wave)

Since the a_n are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad \text{etc.}$$

Their graphs in Figure 4 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and k of our function, at these points. This is typical.

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \frac{\pi}{2}$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

Thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

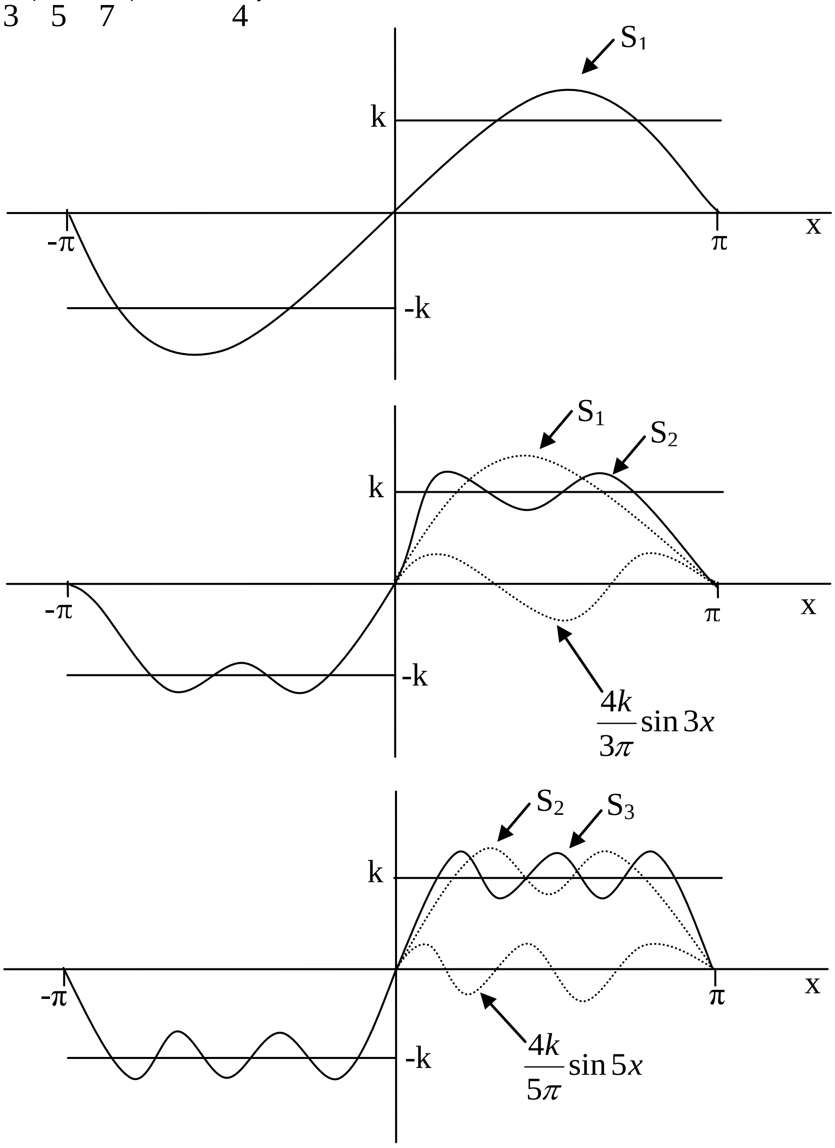


Figure 4 First Three Partial Sums of the Corresponding Fourier Series

1.3 Theorem (Orthogonality of the Trigonometric System)

The trigonometric system (3) is orthogonal on the interval $-\pi \le x \le \pi$ (hence also on $0 \le x \le 2\pi$ or any other interval of length 2π because of periodicity); that is, the integral of the product of any two functions in (3) over that interval is 0, so that for any integers n and m,

$$\text{(a)} \quad \int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \qquad (n \ne m)$$

$$\text{(b)} \quad \int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \ne m) \qquad (8)$$

$$\text{(c)} \quad \int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad (n \ne m \text{ or } n = m)$$

Proof

This follows simply by transforming the integrands trigonometrically from products into sums.

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{(n+m)}\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{\sin(n-m)x}{(n-m)}\right]_{-\pi}^{\pi}.$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx$$

$$= \frac{1}{2}\left[\frac{\sin(n-m)x}{(n-m)}\right]_{-\pi}^{\pi} - \frac{1}{2}\left[\frac{\sin(n+m)x}{(n+m)}\right]_{-\pi}^{\pi}.$$

Since $m \ne n,$ the integrals on the right are all 0.
Similarly, in (8c), for all integer m and n

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x dx.$$

For $m \ne n,$ $\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x dx$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)x}{(n+m)} - \frac{\cos(n-m)x}{(n-m)}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{2(n+m)}\left[\cos(n+m)\pi - \cos(n+m)\pi\right]$$

$$-\frac{1}{2(n-m)}\left[\cos(n-m)\pi - \cos(n-m)\pi\right]$$

$$= 0.$$

For $m = n,\ \int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx dx$

$$= \frac{1}{2}\left[-\frac{\cos 2mx}{2m}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{4m}\left[\cos 2m\pi - \cos 2m\pi\right] = 0.$$

1.4 Derivation of the Euler Formulas

We prove $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$ Integrating on both sides of (5) from $-\pi$ to π, we get $\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx.$

We now assume that termwise integration is allowed. Then we obtain

$$\int_{-\pi}^{\pi} f(x)dx = a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx dx + b_n\int_{-\pi}^{\pi}\sin nx dx\right).$$

The first term on the right equals $2\pi a_0$. Integration shows that all the other integrals are 0. Hence division by 2π gives $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$

We prove $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$. Multiplying (5) on both sides by $\cos mx$ with any fixed positive integer m and integrating from $-\pi$ to π, we have

$$\int_{-\pi}^{\pi} f(x)\cos mx dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx dx. \qquad (9)$$

We now integrate term by term. Then on the right we obtain an integral of $a_0 \cos mx$, which is 0; an integral of $a_n \cos nx \cos mx$, which is $a_m\pi$ for $n = m$ and 0 for $n \neq m$ by (8a); and an integral of $b_n \sin nx \cos mx$, which is 0 for all n and m by (8c).Hence the right side of (9) equals $a_m\pi$. Division by π gives $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx$ (with m instead of n).

We finally prove $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$. Multiplying (5) on both sides by $\sin mx$ with any fixed positive integer m and integrating from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\sin mx dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\sin mx dx.$$

Integrating term by term, we obtain on the right an integral of $a_0 \sin mx$, which is 0; an integral of $a_n \cos nx \sin mx$, which is 0 by (8c); and an integral of $b_n \sin nx \sin mx$, which is $b_m\pi$ if $n = m$ and 0 if $n \neq m$, by (8b). This implies $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx$

(with n denoted by m).This completes the proof of the Euler formulas (6) for the Fourier coefficients.

1.5 Fourier Cosine and Sine Series

If $f(x)$ is an even function, that is, $f(-x) = f(x)$ (see Figure 5), its Fourier series (5) reduces to a Fourier cosine seriess

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \quad (f \text{ even})$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \; a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L} xdx, \; n = 1, 2, ...$$

If $f(x)$ is an odd function, that is, $f(-x) = -f(x)$ (see Figure 6), its Fourier series (5) reduces to a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad (f \text{ odd})$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L} xdx.$$

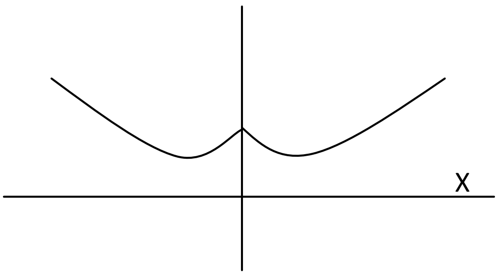


Figure 5 Even Function

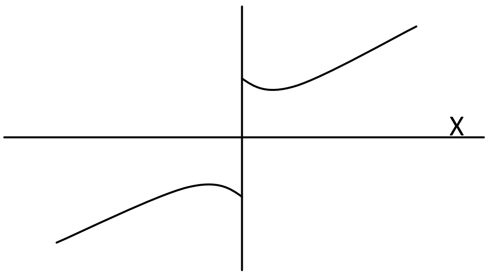


Figure 6 Odd Function

These formulas follow from (5) and (6) by remembering from calculus that the definite integral gives the net area (= area above the axis minus area below the axis) under the curve of a function between the limits of integration. This implies

$$\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx \quad \text{for even } g,$$

$$\int_{-L}^{L} h(x)dx = 0 \quad \text{for odd } h,$$

1.5.1 Example

We will find the two half -range expansions of the function (Figure 7)

$$f(x) = \begin{cases} \dfrac{2k}{L} x & \text{if} \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L - x) & \text{if} \quad \dfrac{L}{2} < x < L \end{cases}$$

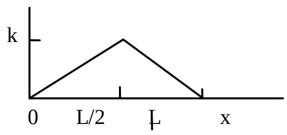


Figure 7 The given Function in Example 1.5.1

(a) Even periodic extension. From

$$a_0 = \frac{1}{L}\int_0^L f(x)dx,\ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx,\ n = 1, 2, ...$$

we obtain

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} xdx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)dx\right] = \frac{k}{2},$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\cos\frac{n\pi}{L}xdx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi}{L}xdx\right].$$

We consider a_n .For the first integral we obtain by integration by parts

$$\int_0^{\frac{L}{2}} x\cos\frac{n\pi}{L}xdx = \frac{Lx}{n\pi}\sin\frac{n\pi}{L}x\Big|_0^{\frac{L}{2}} - \frac{L}{n\pi}\int_0^{\frac{L}{2}}\sin\frac{n\pi}{L}xdx$$

$$= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}(\cos\frac{n\pi}{2} - 1).$$

Similarly, for the second integral we obtain

$$\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi}{L}xdx = \frac{L}{n\pi}(L-x)\sin\frac{n\pi}{L}x\Big|_{\frac{L}{2}}^{L} + \frac{L}{n\pi}\int_{\frac{L}{2}}^{L}\sin\frac{n\pi}{L}xdx$$

$$= \left(0 - \frac{L}{n\pi}\left(L - \frac{L}{2}\right)\sin\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right).$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2),\ a_6 = -16k/(6^2\pi^2),\ a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$ .Hence the first half-range expansion of $f(x)$ is (Figure 8a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) Odd periodic extension. Similarly, from $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}xdx$

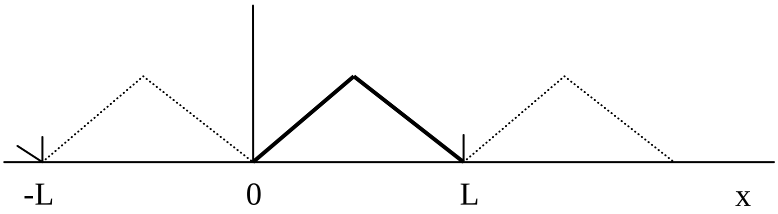
we obtain

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

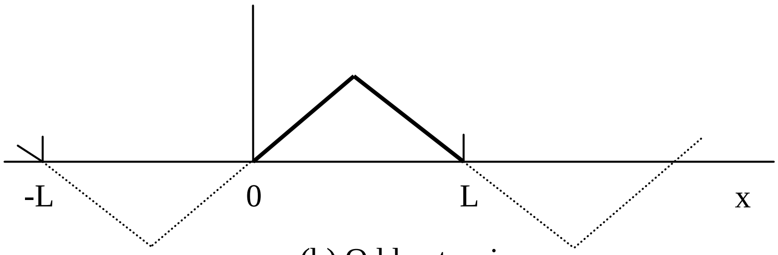
Hence the other half-range expansion of $f(x)$ is (Figure 8b)

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of *f(x)*, of period 2*L*.



(a) Even extension



(b) Odd extension

Figure (8) Periodic Extensions of f(x)

2. Basic Concepts of Partial Differential Equations

A partial differential equation is an equation that involves an unknown function of two (or) more independent variables and certain partial derivatives of the unknown functions. More precisely, let u denote a function of the n independent variables $x_1, x_2, ..., x_n,\ n \geq 2$.

Then a relation of the form

$$f(x_1, x_2, ..., x_n, u, u_{x_1}, u_{x_2}, ..., u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, ...) = 0$$

Where f is a function of its arguments, is a partial differential equation in *u*.

The following equations are some examples of partial differential equations in two independent variables *x* and *y*.

$$xu_x + yu_y - 2u = 0 \tag{10}$$

$$yu_x + xu_y = x \tag{11}$$

$$u_{xx} + u_{yy} = u \tag{12}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = y \tag{13}$$

$$u_{xx} + xu_y^2 + yu = y. \tag{14}$$

2.1 Order of Partial Differential Equation

As in ordinary differential equation, the highest-order derivative appearing in a partial differential equation is called the order of the equation. Thus in the above (10) and (11) are first order partial differential equations and all remaining three equations are second order.

2.2 Linearity

A partial differential equation in the function *u* is said to be linear, if it is at most of first degree in *u* and the derivatives of *u*. This means that the equation should not contain any term that involves powers (or) products of *u* and derivatives of *u*.
Thus, the equation,

$$xu_x + yu_y - 2u = 0 \tag{15}$$

$$yu_x - xu_y = u \tag{16}$$

$$u_{xx} - u_y + u_{yy} = 0 \tag{17}$$

are called linear partial differential equations. On the other hand, the equations

$$uu_x + yu_x + u = xy^2 \tag{18}$$

$$u_{xx} + xu_y^2 + yu = y \tag{19}$$

are not linear because the former involves product of u and u_x, where as the latter involves second power of u_y.

A partial differential equation that is not linear is called a nonlinear partial differential equation.

2.3 Linear Operators

An operator *L* is said to be linear if it satisfies the following.

(i) A constant *c* may be taken outside the operator:

$$L(cu) = cL(u). \tag{20}$$

(ii)The operator operating on the sum of two functions gives the sum of the operating on the individual functions:

$$L(u_1 + u_2) = L(u_1) + L(u_2). \tag{21}$$

We may combine equations (20) and (21) as

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2), \tag{22}$$

where c_1 and c_2 are constants.

2.4 Example

The wave operator:

$$L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$$

is a linear partial differential operator.

To see this we proceed as follows.

$$L(c_1u_1 + c_2u_2) = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)(c_1u_1 + c_2u_2)$$

$$= \frac{\partial^2}{\partial t^2}(c_1u_1 + c_2u_2) - c^2 \frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2)$$

$$= c_1\left[\frac{\partial^2 u_1}{\partial t^2} - c^2\frac{\partial^2 u_1}{\partial x^2}\right] + c_2\left[\frac{\partial^2 u_2}{\partial t^2} - c^2\frac{\partial^2 u_2}{\partial x^2}\right]$$

$$= c_1 L(u_1) + c_2 L(u_2)$$

Thus wave operator is linear.

2.5 Principle of Superposition

An equation of the form

$$Lu = f \tag{23}$$

Where f is a given function, is called a linear partial differential equation.

If $f = 0$, (23) is said to be homogeneous; otherwise, it is called non-homogeneous.

Let $u_1, u_2, ..., u_n$ be *n* functions that satisfy the homogeneous equation

$$Lu = 0, \tag{24}$$

where *L* is a linear partial differential operator.

Then by (22), the linear combination $u = \sum_{i=1}^{n} c_i u_i$, where c_i are constants, is also a solution. This is called the principle of superposition. From Example 2.4, if u_1 and u_2 satisfy the partial differential equation $u_{tt} - c^2 u_{xx} = 0$, then we can show that $u_1 + u_2$ satisfies the given partial differential equation.

To see this we proceed as follows,

$$L.H.S = (u_1 + u_2)_{tt} - c^2(u_1 + u_2)_{xx}$$

$$= \left\{(u_1)_{tt} - c^2(u_1)_{xx}\right\} + \left\{(u_2)_{tt} - c^2(u_2)_{xx}\right\}$$

$$= 0$$

$$= R.H.S.$$

Thus $u_1 + u_2$ satisfies the given partial differential equation and principle of superposition is obeyed.

3. Solution by the Method of Separating Variables

3.1 Vibrating String Problem

The model of a vibrating elastic string (a violin string, for instance) consists of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{25}$$

for the unknown deflection $u(x,t)$ of the string, a partial differential equation that we have just obtained, and some additional conditions, which we shall now derive.

Since the string is fastened at the ends $x = 0$ and $x = L$, we have the two boundary conditions

$$(a) u(0,t)=0,\ (b) u(L,t)=0 \quad \text{for all } t \geq 0. \tag{26}$$

Furthermore, the form of the motion of the string will depend on its initial deflection (deflection at time $t=0$),call it $f(x)$, and on its initial velocity (velocity at time $t=0$), call it $g(x)$. We thus have the two initial conditions

$$(a) u(x,0)=f(x),\ (b) u_t(x,0)=g(x)\ (0 \leq x \leq L) \tag{27}$$

where $u_t = \dfrac{\partial u}{\partial t}$. We now have to find a solution of the partial differential equation (25) satisfying the conditions (26) and (27). This will be the solution of our problem. We shall do this in three steps.

In the method of separating variables, or product method, we determine solutions of the wave equation (25) of the form

$$u(x,t)=F(x)G(t) \tag{28}$$

which are a product of two functions, each depending on only one of the variables *x* and *t*. This is a powerful general method that has various applications in engineering mathematics, as we shall see in this chapter. Differentiating (28), we obtain

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

where dots denote derivatives with respect to *t* and primes derivatives with respect to *x*. By inserting this into the wave equation (25) we have

$$F\ddot{G} = c^2 F''G.$$

Dividing by $c^2 FG$ and simplifying gives

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

The variables are now separated, the left side depending only on *t* and the right side only on *x*. Hence both sides must be constant because, if they were variable, then changing *t* or *x* would affect only one side, leaving the other unaltered. Thus, say,

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k.$$

Multiplying by the denominators gives immediately two ordinary differential equations

$$F'' - kF = 0 \tag{29}$$

and

$$\ddot{G} - c^2 kG = 0. \tag{30}$$

Here, the separation constant *k* is still arbitrary.

We now determine solution *F* and *G* of (29) and (30) so that $u = FG$ satisfies the boundary condition (26) that is,

$$u(0,t)=F(0)G(t)=0,\ u(L,t)=F(L)G(t)=0 \quad \text{for all } t. \tag{31}$$

We first solve (29). If $G \equiv 0,$ then $u = FG \equiv 0,$ which is of no interest. Hence $G \not\equiv 0$ and then by (31),

$$\text{(a) } F(0)=0, \quad \text{(b) } F(L)=0. \tag{32}$$

We show that *k* must be negative. For $k = 0$ the general solution of (29) is $F(x) = ax + b,$ and from (32) we obtain $a = b = 0,$ so that $F \equiv 0$ and $u = FG \equiv 0,$ which is of no interest. For positive $k = \mu^2$ a general solution of (29) is

$$F(x) = Ae^{\mu x} + Be^{-\mu x}$$

and from (32) we obtain $F \equiv 0$. Hence we are left with possibility of choosing *k* negative, say, $k = -p^2.$ Then (29) becomes $F'' + p^2 F = 0$ and has as a general solution

$$F(x) = A\cos px + B\sin px.$$

From this and (32) we have

$$F(0) = A = 0 \text{ and then } F(L) = B\sin pL = 0.$$

We must take $B \neq 0$ since otherwise $F \equiv 0.$ Hence $\sin pL = 0.$ Thus

$$pL = n\pi, \quad \text{so that} \quad p = \frac{n\pi}{L} \quad (n \text{ integer}). \tag{33}$$

Setting $B = 1,$ we thus obtain infinitely many solutions $F(x) = F_n(x),$ where

$$F_n(x) = \sin\frac{n\pi}{L}x \ (n = 1, 2, ...). \tag{34}$$

These solutions satisfy (32). [For negative integer *n* we obtain essentially the same solutions, except for a minus sign, because $\sin(-\alpha) = -\sin(\alpha).$]

We now solve (30) with $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$ resulting from (33), that is,

$$\ddot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$G_n(t) = B_n\cos\lambda_n t + B_n^*\sin\lambda_n t.$$

Hence solutions of (25) satisfying (26) are $u_n(x,t) = F_n(x)G_n(t) = G_n(t)F_n(x),$ written out

$$u_n(x,t) = (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x \ (n = 1, 2, ...). \tag{35}$$

These functions are called the eigenfunctions, or characteristic functions, and the values $\lambda_n = \frac{cn\pi}{L}$ are called the eigenvalues, or characteristic values, of the vibrating string.

Since Equation (25) is linear and homogeneous, by the principle of superposition, the infinite series

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)$$
$$=\sum_{n=1}^{\infty}(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x. \quad (36)$$

Then applying the initial condition (27a),we obtain

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{L}x=f(x).\ (0\le x\le L). \quad (37)$$

Hence we must choose the B_n's so that $u(x,0)$ becomes the Fourier sine series of $f(x)$. Thus, by (28),

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}xdx,\ n=1,2,\ldots. \quad (38)$$

Similarly, by differentiating (36) with respect to t and using (27b), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\left[\sum_{n=1}^{\infty}(-B_n\lambda_n\sin\lambda_n t+B_n^*\lambda_n\cos\lambda_n t)\sin\frac{n\pi}{L}x\right]_{t=0}$$
$$=\sum_{n=1}^{\infty}B_n^*\lambda_n\sin\frac{n\pi}{L}x=g(x).$$

Hence we must choose the B_n^*'s so that for $t=0$ the derivative $\frac{\partial u}{\partial t}$ becomes the Fourier sine series of $g(x)$. Thus, again by (28),

$$B_n^*\lambda_n=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}xdx.$$

Since $\lambda_n=\frac{cn\pi}{L}$, we obtain by division

$$B_n^*=\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}xdx,\ n=1,2,\ldots. \quad (39)$$

Hence, the solution of the vibrating problem is given by the series (36) the coefficients B_n and B_n^* are determined by formulae (38), (39).

3.2 Example

We can find the solution of wave equation (25) satisfying (26) and corresponding to the triangular initial deflection

$$f(x)=\begin{cases}\frac{2k}{L}x & if\quad 0<x<\frac{L}{2}\\ \frac{2k}{L}(L-x) & if\quad \frac{L}{2}<x<L\end{cases}$$

and initial velocity zero.

Since $g(x) \equiv 0,$ we have $B_n^* = 0$ in (36).

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}xdx$$

$$= \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\sin\frac{n\pi}{L}xdx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi}{L}xdx\right].$$

Using the integration by parts, we obtain

$$\frac{L^2}{4k}B_n = \left[\frac{L}{n\pi}x(-\cos\frac{n\pi}{L}x) + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{L}x\right]_0^{\frac{L}{2}}$$

$$+\left[\frac{L}{n\pi}(L-x)(-\cos\frac{n\pi}{L}x) - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{L}x\right]_{\frac{L}{2}}^{L}$$

$$B_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

For n is even, $B_n = 0,\ n = 2, 4, \ldots.$

For $n = 1, 5, 9, \ldots,\ B_n = \frac{8k}{n^2\pi^2}.$

For $n = 3, 7, 11, \ldots,\ B_n = -\frac{8k}{n^2\pi^2}.$

Thus, the solution of the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty}\left[B_n\cos(\frac{n\pi c}{L})t + B_n^*\sin(\frac{n\pi c}{L})t\right]\sin\frac{n\pi}{L}x$$

$$= \sum_{n=1}^{\infty} B_n\sin(\frac{n\pi}{L})x\cos(\frac{n\pi c}{L})t$$

$$= \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin(\frac{\pi}{L})x\cos(\frac{\pi c}{L})t - \frac{1}{3^2}\sin(\frac{3\pi}{L})x\cos(\frac{3\pi c}{L})t + - \cdots\right].$$

Conclusion

In this research paper, separation of variables is one of the simplest methods, and the most widely used method, for solving partial differential equations. The wave equation is the simplest and most important in partial differential equations. Other researchers can also study continuously heat equation and Laplace equation by using separation of variables method.

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